ON GENERALIZED NUMERICAL RANGES
OF QUADRATIC OPERATORS

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ABSTRACT. It is shown that the result of Tso-Wu on the elliptical shape of the numerical range of quadratic operators holds also for the essential numerical range. The latter is described quantitatively, and based on that sufficient conditions are established under which the c-numerical range also is an ellipse. Several examples are considered, including singular integral operators with the Cauchy kernel and composition operators.

1. INTRODUCTION

Let A be a bounded linear operator acting on a complex Hilbert space $\mathcal{H}$. Recall that the *numerical range $W(A)$* of A is defined as

$$W(A) = \{\langle Ax, x \rangle \colon x \in \mathcal{H}, \|x\| = 1\}.$$

If c is a k-tuple of non-zero (in general, complex) numbers $c_1, \dots, c_k$, then the *c-numerical range* of A is

$$W_c(A) = \left\{ \sum_{j=1}^{k} c_j \langle Ax_j, x_j \rangle \colon \{x_j\}_{j=1}^{k} \text{ is an orthonormal subset of } \mathcal{H} \right\}.$$

Of course, if c consists of just one number $c_1 = 1$, $W_c(A)$ is nothing but the regular numerical range of A. Also, for $c_1 = \dots = c_k = 1$, the c-numerical range $W_c(A)$ turns into $W_k(A)$ – the so called *k-numerical range* [1] introduced by Halmos, see [16]. Finally, the *essential numerical range* introduced in [29] can be defined [11] as

$$(1.1) \qquad\qquad W_{\text{ess}}(A) = \bigcap \operatorname{cl} W(A + K),$$

where the intersection is taken over all compact on $\mathcal{H}$ operators K, and the symbol cl denotes the topological closure. Considering $W_c(A)$ or $W_{\text{ess}}(A)$, we will implicitly suppose that $\dim \mathcal{H} \geq k$ or that $\mathcal{H}$ is infinite dimensional, respectively.

There are several monographs devoted to the numerical range and its various generalizations (including those mentioned above), see for example [5, 15]. We mention here only the results which are of direct relevance to the subject of this paper.

From the definitions it is clear that all three sets are unitarily invariant:

1991 *Mathematics Subject Classification.* Primary 47A12; Secondary 45E05, 47B33, 47B35.

Key words and phrases. numerical range, essential numerical range, c-numerical range, quadratic operator, singular integral operator, composition operator.

[1]We realize that there is a slight abuse of notation here, but both $W_c(A)$ and $W_k(A)$ are rather standard, and the meaning is usually clear from the content.

(1.2) $W(U^*AU) = W(A),\ W_c(U^*AU) = W_c(A),\ W_{\mathrm{ess}}(U^*AU) = W_{\mathrm{ess}}(A)$

for any unitary operator U on $\mathcal{H}$. Also, they behave in a nice and predictable way under affine transformations of A:

(1.3) $W(\alpha A + \beta I) = \alpha W(A) + \beta,\ W_{\mathrm{ess}}(\alpha A + \beta I) = \alpha W_{\mathrm{ess}}(A) + \beta,$

and

(1.4) $$W_c(\alpha A + \beta I) = \alpha W_c(A) + \beta \sum_{j=1}^{k} c_j$$

for any $\alpha, \beta \in \mathbb{C}$.

It is a classical result (known as the Hausdorff-Toeplitz theorem) that the set $W(A)$ is convex. Clearly, $W_{\mathrm{ess}}(A)$ is therefore convex as well. The c-numerical range is convex if all c_j lie on the same line passing through the origin but not in general [32]. In what follows, we suppose that c_j satisfy the above mentioned condition. Moreover, since

$$W_c(\alpha A) = W_{\alpha c}(A), \quad \alpha \in \mathbb{C},$$

we then may (and will) without loss of generality suppose that all c_j are real. We will also arrange them in the non-increasing order:

$$c_1 \geq c_2 \ldots \geq c_k,$$

since permutations of c_j leave $W_c(A)$ invariant.

When $\dim \mathcal{H} = 2$, the numerical range of A is the closed (as is always the case in finite dimensional setting) elliptical disc with the foci at the eigenvalues λ_1, λ_2 of A and the minor axis $\sqrt{\mathrm{tr}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2}$ (the elliptic range theorem, see, e.g., [15, Section 1.1]). According to the Cayley - Hamilton theorem, A in this setting satisfies the equation

(1.5) $A^2 - 2\mu A - \nu I = 0$

with

$$\mu = (\lambda_1 + \lambda_2)/2, \quad \nu = -\lambda_1 \lambda_2.$$

For arbitrary $\mathcal{H}$, operators A satisfying (1.5) with some $\mu, \nu \in \mathbb{C}$ are called *quadratic operators*.

Rather recently, Tso and Wu showed that $W(A)$ is an elliptical disc (open or closed) for any quadratic operator A, independent of the dimension of $\mathcal{H}$ [30].

In this paper, we continue considering the (generalized) numerical ranges of quadratic operators. We start by stating Tso-Wu's result and outlining its proof (different from one presented in [30]), in order to show how it can be modified to prove ellipticity of the *essential* numerical ranges of quadratic operators. We then use the combination of the two statements to derive some sufficient conditions for the c-numerical range to also have an elliptical shape. This is all done in Section 1. Section 2 is devoted to concrete implementations of these results.

2. Main results

2.1. **Classical numerical range.** We begin with the Tso-Wu result.

Theorem 2.1. *Let the operator A satisfy equation (1.5). Then $W(A)$ is the elliptical disc with the foci $\lambda_{1,2} = \mu \pm \sqrt{\mu^2 + \nu}$ and the major/minor axis of the length*

$$(2.1) \qquad s \pm \left|\mu^2 + \nu\right| s^{-1}.$$

Here $s = \|A - \mu I\|$, and the set $W(A)$ is closed when the norm $\|A - \mu I\|$ is attained and open otherwise.

Proof. As in [30, Theorem 1.1], observe first that (1.5) guarantees unitary similarity of A to an operator of the form

$$(2.2) \qquad \lambda_1 I \oplus \lambda_2 I \oplus \begin{bmatrix} \lambda_1 I & 2X \\ 0 & \lambda_2 I \end{bmatrix}$$

acting on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus (\mathcal{H}_3 \oplus \mathcal{H}_3)$, where $\dim \mathcal{H}_j \ (\geq 0)$ are defined by A uniquely, and X is a positive definite operator on $\mathcal{H}_3$. According to the first of properties (1.2), we may suppose that A itself is of the form (2.2).

Using the first of formulas (1.3) we may further suppose that $\mu = 0$ and $\nu \geq 0$; in other words, that in (2.2)

$$(2.3) \qquad \lambda_1 = -\lambda_2 := \lambda \geq 0, \quad \lambda^2 = \nu.$$

The case $\mathcal{H}_3 = \{0\}$ corresponds to the normal operator A when $W(A)$ is the closed line segment connecting λ_1 and λ_2. This is in agreement with formula (2.1) when $\nu \neq 0$, since in this case $s = \sqrt{\nu}$ is attained, and $s - \sqrt{\nu}s^{-1} = 0$. In the trivial case $s = 0$ (when the operator A is scalar and $W(A)$ degenerates into a single point) formula (2.1) formally speaking is not valid since s^{-1} is not defined. However, the relation between s and ν justifies the convention $\nu s^{-1} = 0$ in this case.

In the non-trivial case $\dim \mathcal{H}_3 > 0$ our argument is different from that in [30]. Namely, we will make use of the fact that the (directed) distance from the origin to the support line ℓ_θ with the slope θ of $W(A)$ is the maximal point ω_θ of the spectrum of $\mathrm{Re}(ie^{-i\theta}A)$. Moreover, ℓ_θ actually contains points of $W(A)$ if and only if ω_θ belongs to the point spectrum of $\mathrm{Re}(ie^{-i\theta}A)$.

For A of the form (2.2) with λ_j as in (2.3),

$$\mathrm{Re}(ie^{-i\theta}A) = (\lambda \sin \theta)I \oplus (-\lambda \sin \theta)I \oplus \begin{bmatrix} (\lambda \sin \theta)I & ie^{-i\theta}X \\ -ie^{i\theta}X & (-\lambda \sin \theta)I \end{bmatrix}.$$

Thus,

$$(2.4) \quad \mathrm{Re}(ie^{-i\theta}A) - \omega I =$$

$$(\lambda \sin \theta - \omega)I \oplus (-\lambda \sin \theta - \omega)I \oplus \begin{bmatrix} (\lambda \sin \theta - \omega)I & ie^{-i\theta}X \\ -ie^{i\theta}X & -(\lambda \sin \theta + \omega)I \end{bmatrix}.$$

For any $\omega \neq \lambda \sin \theta$, the last direct summand in (2.4) can be rewritten as
$$(2.5)$$
$$\begin{bmatrix} I & 0 \\ 0 & (\lambda \sin \theta - \omega)^{-1}I \end{bmatrix} \begin{bmatrix} I & 0 \\ -ie^{i\theta}X & I \end{bmatrix} \begin{bmatrix} (\lambda \sin \theta - \omega)I & ie^{-i\theta}X \\ 0 & (\omega^2 - \lambda^2 \sin^2 \theta)I - X^2 \end{bmatrix}.$$

Therefore, $\omega_\theta = \sqrt{\lambda^2 \sin^2 \theta + \|X\|^2}$ is the rightmost point of the spectrum of $\mathrm{Re}(ie^{-i\theta}A)$. In other words, the support lines of $W(A)$ are the same as of the

numerical range of the 2×2 matrix

$$\begin{bmatrix} \lambda & 2\,\|X\| \\ 0 & -\lambda \end{bmatrix}.$$

The description of $W(A)$ as the elliptical disc with the foci and axes as given in the statement of the theorem follows from here and the elliptic range theorem.

Moreover, ω_θ is an eigenvalue of $\mathrm{Re}(ie^{-i\theta}A)$ if and only if the norm of X (or equivalently, of A itself) is attained, so that this either happens for all θ or for none of them. In the former case, every support line of $W(A)$ must contain at least one of its points, and the elliptical disc $W(A)$ is closed. In the latter case, the support lines are disjoint with $W(A)$, so that it is open. $\qquad \square$

Remark. Formula (2.1) is formally different from the result of [30, Theorem 2.1], where the lengths of the axes of $W(A)$ are given in terms of $\|A - \lambda_1 I\|$, not $\|A - \mu I\|$. The two operators coincide when $\mu^2 + \nu = 0$. If this is not the case, the relation between their norms follows from the general property

$$\|P\| = \frac{1}{2}(\|S\| + \|S\|^{-1})$$

of any projection P and associated with it involution $S = 2P - I$ (see [27]) applied to $P = (A - \lambda_1 I)/(\lambda_2 - \lambda_1)$ and $S = (A - \mu I)/\sqrt{\mu^2 + \nu}$.

As a matter of fact, the relation between A and involution operators shows that A can be represented as a (rather simple) function of two orthogonal projections. This observation allows to describe the spectra and norms of all operators involved in the proof of Theorem 2.1 straightforwardly, using the machinery developed in [28]. We chose an independent exposition, in the interests of self containment.

2.2. **Essential numerical range.** If A satisfies (1.5) and one of its eigenvalues (say λ_1) has finite multiplicity, then in representation (2.2) the spaces $\mathcal{H}_1$ and $\mathcal{H}_3$ are finite dimensional. Thus, A differs from $\lambda_2 I$ by a compact summand, and $W_{\mathrm{ess}}(A)$ is a single point. Let us exclude this trivial situation, that is, suppose that $\sigma_{\mathrm{ess}}(A) = \sigma(A) = \{\lambda_1, \lambda_2\}$.

From (1.1) it is clear that the support lines $\ell_\theta^{\mathrm{ess}}$ with the slope θ are at the distance $\omega_\theta^{\mathrm{ess}}$ from the origin. Here $\omega_\theta^{\mathrm{ess}}$ is the maximal point of the essential spectrum of $\mathrm{Re}(ie^{-i\theta}A)$. This observation allows to repeat the statement and the proof of Theorem 2.1 almost literally, inserting the word "essential" where appropriate (of course, the last paragraph of the proof becomes irrelevant since the essential numerical range is always closed). We arrive at the following statement.

Theorem 2.2. *Let the operator A satisfy equation* (1.5), *with both eigenvalues $\lambda_{1,2} = \mu \pm \sqrt{\mu^2 + \nu}$ having infinite multiplicity. Then $W_{\mathrm{ess}}(A)$ is the closed elliptical disc with the foci $\lambda_{1,2}$ and the major/minor axis of the length $s_0 \pm \left|\mu^2 + \nu\right| s_0^{-1}$, where s_0 is the essential norm of $A - \mu I$.*

In the trivial case $s_0 = 0$ (when A differs from μI by a compact summand, so that necessarily $\mu^2 + \nu = 0$) we by convention set $\left|\mu^2 + \nu\right| s_0^{-1} = 0$. This agrees with the fact that $W_{\mathrm{ess}}(A)$ then degenerates into a singleton μ.

Corollary 2.3. *Let the operator A satisfying* (1.5) *be such that*

(2.6) $$\|A - \mu I\| > \|A - \mu I\|_{\mathrm{ess}}.$$

Then the elliptical disc $W(A)$ is closed.

Proof. Indeed, (2.6) holds if and only if $\|X\|_{\mathrm{ess}} < \|X\|$ for X from (2.2). Being positive definite, the operator X then has $\|X\|$ as its eigenvalue. In other words, the norm of X (and therefore of $A - \mu I$) is attained. It remains to invoke the last statement of Theorem 2.1. $\qquad\square$

2.3. **c-numerical range.** The behavior of $W_c(A)$, even for quadratic operators, is more complicated; see [8] for some observations on the k-numerical range. With no additional assumptions on A, we give only a rather weak estimate. In what follows, it is convenient to use the notation $\|c\| = \sum_{j=1}^{k} |c_j|$.

Lemma 2.4. *Let A be as in Theorem 2.2. Denote by s and s_0 the norm and essential norm of $A - \mu I$ respectively, and by E and E_0 two elliptical discs with the foci at $\mu \sum_{j=1}^{k} c_j \pm \sqrt{\mu^2 + \nu}\, \|c\|$, the first – closed, with the axes $(s \pm |\mu^2 + \nu|\, s^{-1})\, \|c\|$ and the second – open, with the axes $(s_0 \pm |\mu^2 + \nu|\, s_0^{-1})\, \|c\|$. Then $W_c(A)$ contains E_0 and is contained in E.*

Proof. Using (1.4) we may, as in the proof of Theorem 2.1, without loss of generality suppose that $\mu = 0$, $\nu \geq 0$. Since all the sets E, E_0 and $W_c(A)$ are convex, we need only to show that the support line to $W_c(A)$ in any direction lies between the respective support lines to E_0 and E. In other words, the quantity

$$(2.7) \qquad \sup \left\{ \sum_{j=1}^{k} c_j \operatorname{Re}\langle ie^{-i\theta} A x_j, x_j \rangle \colon \{x_j\}_{j=1}^{k} \text{ is orthonormal} \right\}$$

must lie between

$$\|c\|\sqrt{\nu \sin^2 \theta + \|X\|_{\mathrm{ess}}^2} \quad \text{and} \quad \|c\|\sqrt{\nu \sin^2 \theta + \|X\|^2}$$

with X given by (2.2). But this is indeed so, because (2.5) implies that the spectrum and the essential spectrum of $\operatorname{Re}(ie^{-i\theta} A)$ have the endpoints $\pm\sqrt{\nu \sin^2 \theta + \|X\|^2}$ and $\pm\sqrt{\nu \sin^2 \theta + \|X\|_{\mathrm{ess}}^2}$, respectively. $\qquad\square$

An interesting situation occurs when the norm of $A - \mu I$ coincides with its essential norm (equivalently, $\|X\| = \|X\|_{\mathrm{ess}}$ for X from (2.2)), so that E is simply the closure of E_0. To state the explicit result, denote by $m_\pm$ the number of positive/negative coefficients c_j and let $m = \max\{m_+, m_-\}$.

Theorem 2.5. *Let A be as in Theorem 2.2, and on top of that*

$$(2.8) \qquad\qquad\qquad \|A - \mu I\| = \|A - \mu I\|_{\mathrm{ess}}.$$

Define E and E_0 as in Lemma 2.4. Then $W_c(A)$ coincides with E if the norm of $A - \mu I$ is attained on the subspace of the dimension at least m, and with E_0 otherwise.

Proof. Consider first a simpler case when in (2.2) $\dim \mathcal{H}_3 < \infty$. Then due to (2.8) $\mathcal{H}_3 = \{0\}$, so that the operator A is normal. The norm $|\mu^2 + \nu|^{1/2}$ of $A - \mu I$ is attained on infinite dimensional subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and $W_c(A)$ is the closed line segment connecting the points $\mu \sum_{j=1}^{k} c_j + \sqrt{\mu^2 + \nu}\, \|c\|$ and $\mu \sum_{j=1}^{k} c_j - \sqrt{\mu^2 + \nu}\, \|c\|$. This segment apparently coincides with E.

Let now $\mathcal{H}_3$ be infinitely dimensional. From Lemma 2.4 it follows that $W_c(A)$ lies between E and its interior E_0, so that the only question is which points of

the boundary of E belong to $W_c(A)$. It follows from (2.5) that the minimal and maximal points of the spectrum of $\mathrm{Re}(ie^{-i\theta}A)$ have the same multiplicity as its eigenvalues, this multiplicity does not depend on θ and coincides in fact with the dimension d (≥ 0) of the subspace on which the norm of X is attained. From (2.2) under conditions (2.3) it follows that the norm of $A - \mu I$ is attained on a d-dimensional subspace as well.

On the other hand, the supremum in (2.7) is attained if and only if this multiplicity is at least m. Thus, the boundary of E belongs to $W_c(A)$ if $d \geq m$ and is disjoint with $W_c(A)$ otherwise. □

3. Examples

We consider here several concrete examples illustrating the above stated abstract results. All the operators A involved happen to be involutions which corresponds to the choice $\mu = 0$, $\nu = 1$ in (1.5). According to Theorems 2.1 and 2.2, the major/minor axes of the elliptical discs $W(A)$ and $W_{\mathrm{ess}}(A)$ then have the lengths

$$(3.1) \qquad \|A\| \pm \|A\|^{-1} \;\; \text{and} \;\; \|A\|_{\mathrm{ess}} \pm \|A\|_{\mathrm{ess}}^{-1},$$

respectively.

3.1. **Singular integral operators on closed curves.** Let Γ be the union of finitely many simple Jordan rectifiable curves. Suppose that the number of its points of self-intersection is finite, and that Γ partitions the extended complex plane $\dot{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ into two open disjoint (not necessarily connected) sets D^+ and D^-. Moreover, we suppose that Γ is the common boundary of D^+ and D^-, and that it is oriented in such a way that the points of $D^{\pm}$ lie to the left/right of Γ.

The singular integral operator S with the Cauchy kernel is defined by

$$(3.2) \qquad (S\phi)(t) = \frac{1}{\pi i} \oint_{\Gamma} \phi(\tau) \frac{d\tau}{\tau - t}.$$

It acts as an involution [13] on the linear manifold of all rational functions with the poles off Γ, dense in the Hilbert space $\mathcal{H} = L^2(\Gamma)$, with respect to the Lebesque measure on Γ. This operator is bounded in L^2 norm, and can therefore be continued to the involution acting on the whole $L^2(\Gamma)$, if and only if Γ is the so called *Carleson curve*. This result, along with the definition of Carleson curves, as well as detailed proofs and the history of the subject, can be found in [6]. For our purposes it suffices to know that S is a bounded involution when the curve Γ is piecewise smooth, i.e., admits a piecewise continuously differentiable parametrization.

If Γ is a circle or a line, then S is in fact selfadjoint, and both its norm and essential norm are equal to 1. This situation is trivial from our point of view, since $W(S)$ and $W_{\mathrm{ess}}(S)$ then coincide with the closed interval $[-1, 1]$ and $W_c(S)$ is $[-\|c\|, \|c\|]$.

As it happens [18], circles and lines are the only simple closed curves in $\dot{\mathbb{C}}$ for which S is selfadjoint. On the other hand, for all smooth simple closed curves the essential norm of S is the same, that is, equal to 1 (see [13, Chapter 7] for Lyapunov curves; the validity of the result for general smooth curves rests on the compactness result from [14] and is well known within singular integral community). Thus, lines and circles are the only smooth closed curves in $\dot{\mathbb{C}}$ for which the norm and the essential norm of S coincide. However, such a coincidence is possible for other *piecewise* smooth (even simple) curves.

One such case occurs when Γ is a bundle of m lines passing through a common point, or of m circles passing through two common points. According to [12], then

$$\|S\| = \|S\|_{\mathrm{ess}} \geq \cot \frac{\pi}{4m},$$

with the last inequality turning into equality for at least $m = 1, 2, 3$. Respectively, for such curves Γ the sets $W(S)$, $W_{\mathrm{ess}}(S)$ are the ellipses with the foci at ± 1, coinciding up to the boundary, and with the major axes of the length at least $2 \csc \frac{\pi}{2m}$. This length equals $2 \csc \frac{\pi}{2m}$ for $m = 2, 3$. The c-numerical range of S is the same ellipse, only scaled by $\|c\|$.

The equality $\|S\| = \|S\|_{\mathrm{ess}}$ also holds for Γ consisting of circular arcs (one of which can degenerate into a line segment) connecting the same two points in $\mathbb{C}$ [3, 4]; in order for an appropriate orientation on Γ to exist the number of these arcs must be even. If, in particular, there are two of them (that is, the curve Γ is simple), then

$$\|S\| = \|S\|_{\mathrm{ess}} = D_\phi + \sqrt{D_\phi^2 + 1},$$

where

$$D_\phi = \sup \left\{ \frac{\sinh(\pi \phi \xi)}{\cosh(\pi \xi)} : \xi \geq 0 \right\}$$

and $\pi(1-\phi)$ is the angle between the arcs forming Γ [3]. The ellipses $W(S)$, $W_{\mathrm{ess}}(S)$ therefore have the major axes of the length $2\sqrt{D_\phi^2 + 1}$.

For some particular values of ϕ the explicit value of D_ϕ can be easily computed, see [3]. If, for instance, Γ consists of a half circle and its diameter, that is $\phi = 1/2$, then $D_\phi = 1/2\sqrt{2}$. Respectively, the major axes of $W(S)$ and $W_{\mathrm{ess}}(S)$ have the length $3/\sqrt{2}$.

It would be interesting to describe all curves Γ for which the norm and the essential norm of the operator (3.2) are the same.

3.2. **Singular integral operators on weighted spaces on the circle.** Let now Γ be the unit circle $\mathbb{T}$. We again consider the involution (3.2), this time with $\mathcal{H}$ being the *weighted* Lebesgue space L_ρ^2. The norm on this space is defined by

$$\|f\|_{L_\rho^2} = \|\rho f\|_{L^2} := \frac{1}{\sqrt{2\pi}} \left(\int_0^{2\pi} |f(e^{i\theta})|^2 (\rho(e^{i\theta})^2 d\theta \right)^{1/2},$$

where the weight ρ is an a.e. positive measurable and square integrable function on $\mathbb{T}$. In this setting, the operator S is closely related with the Toeplitz and Hankel operators on Hardy spaces, weighted or not. All needed definitions and "named" results used below and not supplied with explicit references conveniently can be found in the exhaustive recent monograph [23].

3.2.1. Involution S is bounded on L_ρ^2 if and only if ρ^2 satisfies the Helson-Szegő condition, that is, can be represented as

(3.3) $\exp(\xi + \overline{\eta})$ with $\xi, \eta \in L^\infty(\mathbb{T})$ real valued and $\|\eta\|_\infty < \pi/2$

[23, p. 419]. This condition is equivalent to

(3.4) $$\|H_\omega\| < 1,$$

where

(3.5) $$\omega = \overline{\rho_+}/\rho_+,$$

ρ_+ is the outer function such that $|\rho_+| = \rho$ a.e. on $\mathbb{T}$, and H_ω denotes the Hankel operator H_ω with the symbol ω acting from the (unweighted) Hardy space H^2 to its orthogonal complement in L^2. It is also equivalent to invertibility of the Toeplitz operator T_ω on H^2. Moreover [10],

$$\|S\|_{L^2_\rho} = \sqrt{\frac{1 + \|H_\omega\|}{1 - \|H_\omega\|}},$$

and a similar relation holds for the essential norms of S and H_ω. But

$$\|H_\omega\| = \mathrm{dist}(\omega, H^\infty)$$

(Nehari theorem [23, p. 3]) and

$$\|H_\omega\|_{\mathrm{ess}} = \mathrm{dist}(\omega, H^\infty + C)$$

(Adamyan-Arov-Krein theorem [23, Theorem 1.5.3]), where H^∞ is the Hardy class of bounded analytic in $\mathbb{D}$ functions, and its sum with the set C of continuous on $\mathbb{T}$ functions is the *Douglas algebra* $H^\infty + C$. Thus, the ellipses $W(S)$ and $W_{\mathrm{ess}}(S)$ have the major axes

$$2/\sqrt{1 - \mathrm{dist}(\omega, H^\infty)} \text{ and } 2/\sqrt{1 - \mathrm{dist}(\omega, H^\infty + C)},$$

respectively.

The norm of S is attained only simultaneously with the norm of H_ω. This happens, in particular, if H_ω is compact, that is $\omega \in H^\infty + C$. The latter condition can be restated directly in terms of ρ [10] and means that $\log \rho \in VMO$, where VMO (the class of functions with vanishing mean oscillation) is the sum of C with its harmonic conjugate $\widetilde{C}$.

Thus, for all the weights ρ such that $\log \rho \in VMO$ the ellipse $W(S)$ is closed, while $W_{\mathrm{ess}}(S)$ degenerates into the line interval $[-1, 1]$.

A criterion for the norm of H_ω to be attained also can be given, though in less explicit form. Recall that the distance from ω to H^∞ is always attained on some $g \in H^\infty$ (this is part of Nehari theorem). This g in general is not unique, and any f of the form $\omega - g$ is called a *minifunction*. By (another) Adamyan-Arov-Krein's theorem [23, Theorem 1.1.4], the norm of H_ω is attained if and only if the minifunction is unique and can be represented in the form

$$(3.6) \qquad\qquad f(z) = \|H_\omega\| \, \overline{z\theta h}/h,$$

where θ and $h \, (\in H^2)$ are some inner and outer functions of z, respectively [2].

3.2.2. We now turn to possible realizations of the outlined possibilities. If f admits a representation (3.6) with θ of an infinite degree (that is, being an infinite Blaschke product or containing a non-trivial singular factor), then $\|H_\omega\|$ is the s-number of H_ω having infinite multiplicity. In particular,

$$(3.7) \qquad\qquad \|H_\omega\| = \|H_\omega\|_{\mathrm{ess}}.$$

According to Theorem 2.5, $W(S)$ in this case coincides with the closed ellipse $W_{\mathrm{ess}}(S)$, all c-numerical ranges also are closed and differ from $W(S)$ only by an appropriate scaling.

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[2]Formally speaking, Theorem 1.1.4 in [23] contains only the "only if" part. The "if" direction is trivial, since the norm of H_ω is attained on h from (3.6); see Theorem 2.1 of the original paper [2].

Now let θ in (3.6) be a finite Blaschke product of degree $b\,(\geq 0)$ while h is invertible in H^2. Suppose also that $|h|^2$ does not satisfy Helson-Szegő condition, that is, cannot be represented in the form (3.3) (such outer functions are easy to construct – take for example h with $|h|^{\pm 1} \in L^2$ but $|h| \notin L^{2+\epsilon}$ for any $\epsilon > 0$). Then the Toeplitz operator T_f has $(b + 1)$-dimensional kernel, dense (but not closed) range [20, Corollary 3.1 and Theorem 3.16], and therefore is not left Fredholm. By Douglas-Sarason theorem [23, Theorem 1.1.15]

$$\mathrm{dist}(f, H^\infty + C) = |f| = \|H_\omega\| = \|H_f\|\,.$$

We conclude that (3.7) holds again. So, the ellipse $W(S)$ is closed and coincides with $W_{\mathrm{ess}}(S)$. According to Theorem 2.5, the c-numerical range of S is closed if the number of coefficients c_j of the same sign does not exceed $b+1$, and open otherwise.

Finally, if a unimodular function ω is such that the operator T_ω is invertible, (3.7) holds, but its minifunction is not constant a.e. in absolute value, then the norm of H_ω is not attained. Accordingly, all c-numerical ranges, $W(S)$ in particular, in this case are open.

A concrete realization of the latter possibility is given in the next subsection. All the other possibilities mentioned earlier also occur. To construct the respective weights ρ, the following procedure can be applied. Starting with any inner function θ and outer function $h \in H^2$, choose f as in (3.6) with $\|H_\omega\|$ changed to an arbitrary constant in $(0, 1)$. Let ω be an 1-canonical function [3] of the Nehari problem corresponding to the Hankel operator H_f. As such, ω is unimodular, and can be represented as $\omega = g/\overline{g}$, where g is an outer function in H^2 [23, Theorem 5.1.8]. Since $\|H_\omega\| < 1$, the Toeplitz operator $T_{\omega^{-1}}$ is invertible [23, Theorem 5.1.10] (the last two cited theorems from [23] are again by Adamyan-Arov-Krein [2]). The desired weight is given by $\rho = |g|$.

By Treil's theorem [23, Theorem 12.8.1], any positive semi-definite noninvertible operator with zero or infinite dimensional kernel is unitarily similar to the modulus of a Hankel operator. Thus, the multiplicity of the norm of H_ω as its singular value can indeed assume any prescribed value, whether or not (3.7) holds.

3.2.3. Consider the concrete case of *power weights*

(3.8) $$\rho(t) = \prod |t - t_j|^{\beta_j}\,, \quad t_j \in \mathbb{T}, \ \beta_j \in \mathbb{R} \setminus \{0\}.$$

It is an old and well known result that S is bounded on L_ρ^2 with ρ given by (3.8) if and only if $|\beta_j| < 1/2$. This fact, along with other results about such weights cited and used below (and established by Krupnik-Verbitskii [31]) can be found in the monograph [19, Section 5].

The essential norm of S does not depend on the distribution of the nodes t_j along $\mathbb{T}$, and equals

(3.9) $$\|S\|_{\mathrm{ess}} = \cot \frac{\pi(1 - 2\tilde\beta)}{4}, \ \text{where } \tilde\beta = \max |\beta_j|\,.$$

In case of only one node (say t_0, with the corresponding exponent β_0), the norm of S is the same as (3.9). The function ω constructed by this weight ρ in accordance with (3.5) is simply $\omega(t) = t^{\beta_0}$, having a discontinuity at t_0. The distance from ω to H^∞ is the same as to $H^\infty + C$, it equals $\sin(\pi |\beta_0|)$ and is attained on a constant $\ell = \cos(\pi |\beta_0|)e^{i\pi\beta_0}$. A corresponding minifunction $f = \omega - \ell$ is not constant a.e. in

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[3]See [23, p. 156] for the definition.

absolute value; thus, it cannot admit representation (3.6). Consequently, the norm of H_ω is not attained. Accordingly, $W_c(S)$ is open for all c; the numerical range $W(S)$ has the major axis of the length $2\sec(\pi|\beta_0|)$. Other c-numerical ranges are scaled by $\|c\|$, as usual.

More generally, the norm of S coincides with (3.9) independently on the number of nodes, provided that one of the exponents (say β_0) differs by its sign from all others and at the same time exceeds or equals their sum by absolute value. The size and the shape of all the ellipses $W(S)$, W_{ess}, $W_c(S)$ is then the same as for the weight with only one exponent β_0.

In case of two nodes (t_1 and t_2), the condition above holds if the respective exponents β_1, β_2 are of the opposite sign. If the signs are the same, the norm of S actually depends on $\arg t_1/t_2$. It takes its minimal value (for fixed β_j) when $t_1/t_2 < 0$. This value coincides with (3.9), thus making Theorem 2.5 applicable again.

3.3. Composition operators.
For an analytic mapping of the unit disc $\mathbb{D}$ into itself, the *composition operator* C_ϕ is defined as

$$(C_\phi f)(z) = f(\phi(z)).$$

3.3.1.   We consider this operator first on the Hardy space H^2. In this setting, the operator C_ϕ is bounded and, if ϕ is an inner function,

$$(3.10) \qquad \|C_\phi\| = \sqrt{\frac{1 + |\phi(0)|}{1 - |\phi(0)|}},$$

see [22], also [9]. It is easily seen from the proof of (3.10) given there that the norm of C_ϕ is not attained, unless $\phi(0) = 0$. As was shown in [25, 26], the essential norm of C_ϕ coincides with its norm; moreover, this property is characteristic for inner functions.

The numerical ranges of composition operators C_ϕ with ϕ being conformal automorphisms of $\mathbb{D}$ where treated in [7]. It was observed there, in particular, that $W(C_\phi)$ is an elliptical disc with the foci at ± 1 when C_ϕ is an involution, that is,

$$(3.11) \qquad \phi(z) = \frac{p - z}{1 - \overline{p}z}$$

for some fixed $p \in \mathbb{D}$. The major axis of this disc E_p was computed in [1], where as a result of rather lengthy computations it was shown to equal $2/\sqrt{1 - |p|^2}$. For $p = 0$, C_ϕ is an involution of norm 1. Respectively, E_0 degenerates into the closed interval $[-1, 1]$. The question of openness or closedness of E_p for $p \neq 0$ was not discussed.

It follows from Theorem 2.1 that E_p is open (if $p \neq 0$); moreover, the length of its axes can be immediately seen from (3.1) and (3.10):

$$\sqrt{\frac{1 + |p|}{1 - |p|}} + \sqrt{\frac{1 - |p|}{1 + |p|}} = 2/\sqrt{1 - |p|^2}.$$

Furthermore, Theorem 2.2 implies that $W_{\text{ess}}(C_\phi)$ is the closure of E_p. Finally, by Theorem 2.5 the c-numerical range of C_ϕ is E_p dilated by $\|c\|$.

3.3.2.   These results, with some natural modifications, extend to the case of *weighted* spaces H^2_ρ. Namely, for a non-negative function $\rho \in L^2(\mathbb{T})$ with $\log \rho \in L^1$ we define the outer function ρ_+ as in (3.5). Then

$$H^2_\rho = \{f \colon \rho_+ f \in H^2\} \text{ and } \|f\|^2_{H^2_\rho} = \|\rho_+ f\|_{H^2}.$$

A change-of-variable argument, similar to that used in [22], shows the following equality:

$$(3.12) \quad \|C_\phi f\|^2_{H^2_\rho} = \frac{1}{2\pi} \int_0^{2\pi} |f\left(\phi(e^{i\theta})\right)|^2 (\rho(e^{i\theta}))^2 d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\mu})|^2 \left(\rho\left(\phi\left(e^{i\mu}\right)\right)\right)^2 \frac{1-|p|^2}{|p - e^{i\mu}|^2} d\mu = \|f\chi\|^2_{H^2_\rho},$$

where

$$\chi(t) := \frac{\sqrt{1-|p|^2}}{|p - t|} \frac{\rho(\phi(t))}{\rho(t)}, \quad t \in \mathbb{T}.$$

The norm of a multiplication operator on weighted and unweighted Hardy spaces is the same. According to (3.12) the operator C_ϕ is therefore bounded on H^2_ρ if and only if

$$(3.13) \qquad\qquad \sup_{t\in\mathbb{T}} \frac{\rho(\phi(t))}{\rho(t)} < \infty.$$

Observe that (3.13) is equivalent to

$$\inf_{t\in\mathbb{T}} \frac{\rho(\phi(t))}{\rho(t)} > 0$$

because ϕ is an involution. Apparently, (3.13) holds if $\rho \in L^\infty$ is bounded below from 0, but there are plenty of unbounded weights ρ satisfying (3.13) as well.

Under this condition, $\|C_\phi\|_{H^2_\rho} = M$, where

$$(3.14) \qquad\qquad M = \sqrt{1-|p|^2} \sup_{t\in\mathbb{T}} \frac{\rho(\phi(t))}{|p - t|\,\rho(t)}.$$

For any $\epsilon > 0$, consider a function $g \in H^2_\rho$ with the norm one and such that $\|C_\phi g\|_{H^2_\rho} > M - \epsilon$. Then $\|C_\phi g_n\|_{H^2_\rho} > M - \epsilon$ for $g_n(z) = z^n g(z)$, $n = 1, 2, \ldots$. Since the sequence g_n converges weakly to zero in H^2_ρ, from here it follows that the essential norm of C_ϕ also equals M. (We use here the well-known fact that compact operators on Hilbert spaces map weakly convergent sequences into strongly convergent sequences, see [24, Section 85], for example.) Moreover, the norm of C_ϕ is attained if and only if there exist non-zero functions in H^2_ρ with absolute value equal zero a.e. on the subset of $\mathbb{T}$ where $|\chi(t)| \neq M$. Due to uniqueness theorem for analytic functions, a necessary and sufficient condition for this to happen is

$$(3.15) \qquad\qquad \left| \frac{\rho(\phi(t))}{(p - t)\rho(t)} \right| = \text{const a.e. on } \mathbb{T}.$$

If (3.15) holds, then the norm is attained in particular on all inner functions, so that the respective subspace is infinitely dimensional. Consequently, $W_{\text{ess}}(C_\phi)$ is the closed ellipse with the foci at ± 1 and the axes $M \pm M^{-1}$, and $W(C_\phi)$ is the same ellipse when (3.15) holds or its interior when it does not. The c-numerical range is simply $\|c\|\, W(C_\phi)$.

Of course, for $\rho(t) \equiv t$ condition (3.13) holds, formula (3.14) turns into (3.10), and (3.15) is equivalent to $p = 0$. Thus, the results obtained match those already known in the unweighted setting.

3.3.3. One can also consider composition operators C_ϕ on weighted *Lebesgue* spaces L_ρ^2. Formula for the norm and the essential norm of C_ϕ remain exactly the same, with no changes in their derivation[4]. Condition for the norm to be attained is different: in place of (3.15) it is required that the supremum in its left hand side is attained on a set of positive measure. The respective changes in the statement about the numerical ranges are evident, and we skip them. We note only that for $\rho(t) \equiv t$ the supremum in the right hand side of (3.15) either is attained everywhere (if $p = 0$) or just at one point (if $p \neq 0$). Thus, all the sets $W(C_\phi)$, $W_{\text{ess}}(C_\phi)$ and $W_c(C_\phi)$ are exactly the same whether the composition operator C_ϕ with the symbol (3.11) acts on H^2 or L^2.

3.3.4. Finally, we consider the operator C_ϕ on the Dirichlet space $\mathcal{D}$. Recall that the latter is defined as the set of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty,$$

where dA is the area measure.

It was shown in [21, Theorem 2] that for any univalent mapping ϕ of $\mathbb{D}$ onto its subset of full measure,

$$\|C_\phi\|_{\mathcal{D}} = \sqrt{\frac{L + 2 + \sqrt{L(4+L)}}{2}},$$

where $L = -\log(1 - |\phi(0)|^2)$. This simplifies to

$$\|C_\phi\|_{\mathcal{D}} = \frac{\sqrt{L} + \sqrt{4+L}}{2},$$

and is of course applicable when ϕ is given by (3.11). Consequently, the elliptical disc $W(C_\phi)$ has the major axis

$$\sqrt{4 + \log \frac{1}{1 - |p|^2}}.$$

Moreover, the operators considered in [21, Theorem 2] attain their norms, so that $W(C_\phi)$ is closed.

It was further observed in [17, Proposition 2.4] that the essential norm of C_ϕ on $\mathcal{D}$ does not exceed 1, for any univalent ϕ. For ϕ given by (3.11), the essential norm of C_ϕ on $\mathcal{D}$ must be equal 1, since the essential norm of an involution on an infinite dimensional space is at least one. Thus, $W_{\text{ess}}(C_\phi)$ in this setting is the closed interval $[-1, 1]$.

Analogous remarks can be made in other contexts where the norms and essential norms of composition operators are known.

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[4]Moreover, condition $\log \rho \in L^1$ can be weakened simply to ρ being positive a.e. on $\mathbb{T}$, as was the case in Subsection 3.2.

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